Algebra: Simultaneous Equations and Inequalities

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Objectives

- Understand what questions are asking!
- Visualise inequalities.
- Do many simultaneous examples as they should be easy marks
- Get familiar with algebraic fractions
- Use the $-b$ formula yet again!

Review

Simultaneous linear equations

- If have one unknown variable x, need only one equation to solve for x. $5x - 9 = 3$ is all information needed to find x
- If have two unknown variables, x and y, if told

$$x + y = 1000$$

  not enough information. Infinite solution set. A second equation relating x and y is needed to isolate a unique solution
- Usual exam situation is three unknowns x,y and z which now requires three equations relating x,y and z
- Linear means no x^2 or y^2 or xy terms. xy is a cross term. The **cross terms**can be the treacherous ones. If they are absent, all works like clockwork
- **Bonus**: NO CROSS TERMS in circle problems in Paper 2, skills mastered here answer two questions in coordinate geometry

Algebraic Inequalities

- Just like JCH number line problems
- Beware multiplying across an inequality by an algebraic expression that could be negative
- For quadratics, inequality reduces to finding the roots and inequality furnishes domain
- Having a visual image of the problem greatly assists solution.

Skills: Factorising in general

Factorising, for LCH can mean simplifying. Best with examples. Maybe the most challenging? All **Paper 1**

2009 Find the value of $\frac{x}{y}$ when $\frac{2x+3y}{x+6y} = \frac{4}{5}$
Solution Divide through by y and solve.

2008 Simplify fully

$$\frac{x^2+4}{x^2-4} - \frac{x}{x+2}$$

Algebraic Inequalities

2018 Solve the inequality, $\frac{2x-3}{x+2} \geq 3$ given that $x \neq -2$.

2005 Solve $|x - 1| \leq 7$

LCH required skills

Factors, multiples. Difference of squares:$x - y = (\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})$! Difference of cubes. Conditions on roots of quadratic to be real distinct, real equal or complex numbers. Finding common denominators for symbols rather than numbers

Simultaneous Equations 2018

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Solve the simultaneous equations

$$2x + 3y - z = -4$$

$$3x + 2y + 2z = 14$$

$$x - 3z = -13$$

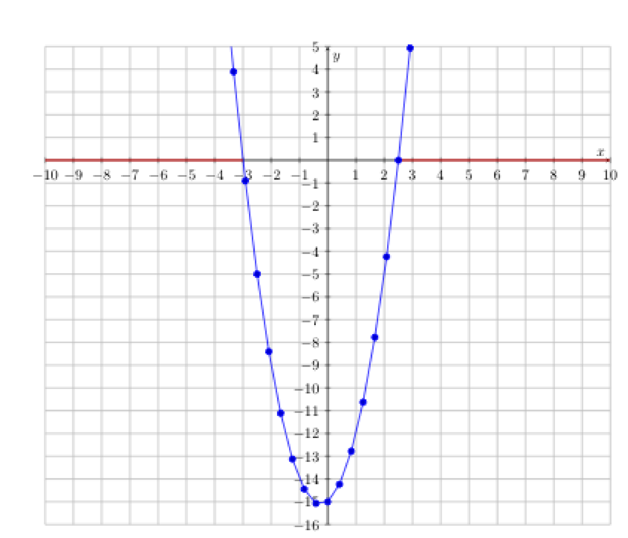
Solution Easier to let $x = 3z - 13$ and substitute, to solve but try yourself.

Quadratic Inequalities 2014

Find values of x such that

$$2x^2 + x - 15 \geq 0$$

Solution It helps to visualise this. We can factorize by inspection or use the $-b$ formula(try yourself). So $x = 2.5$ and $x = -3$ are the roots as we can see in the graph below. Values of x are in red. Why?



Further LCH Problems

2005 The cubic has one integer root and two irrational roots

$$4x^3 + 10x^2 - 7x - 3 = 0$$

Express the two irrational roots in *surd* form.

2006. Solve the simultaneous equations. Note not linear. More than one solution

$$y = 2x - 5$$

and

$$x^2 + xy = 2$$

Useful Vocabulary

- (n.) sign → + or −
- (n.) equation → *something* $= 0$
- (n.) factor → two multiplied factors give result
- (v.) factorise → putting into brackets
- (n.) coefficient → a constant number i.e. a, b, c,d in a pattern $ax^3 + bx^2 + cx + d$
- (n.) cubic → $f(x) = ax^3 + bx^2 + cx + d$
- (n.) root → solution of a quadratic or cubic

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Algebra: Quadratic Functions

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Objectives

- Understand the concepts of roots and factors.
- Revision from JCH
- Factorising Quadratics.
- Proving some useful identities.
- Use conditions on $b^2 - 4ac$ to classify roots.

Quick Revision

Forms of Quadratic Function

- $f(x) = ax^2 + bx + c$ is form in the **mathematical tables**.
- $f(x) = a(x - x_1)(x - x_2)$ is called the **factored form**, where x_1 and x_2 are the roots of the quadratic function. These are the values of x where we cut the x axis.
- $f(x) = a(x - h)^2 + k$ is called the **vertex form**. This is frequently examined as we can quickly determine the local maximum or minimum using the derivative of $f(x)$.

The expression $\Delta = \mathbf{b}^2 - 4ac$

This is the expression under the square root in the $-b$ formula. It tells us how many **real** solutions the quadratic equation has:

$$\text{number of solutions} = \begin{cases} 2 & \text{when } \Delta > 0 \\ 1 & \text{when } \Delta = 0 \\ 0 & \text{when } \Delta < 0 \end{cases}$$

The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Graph of Quadratic Function

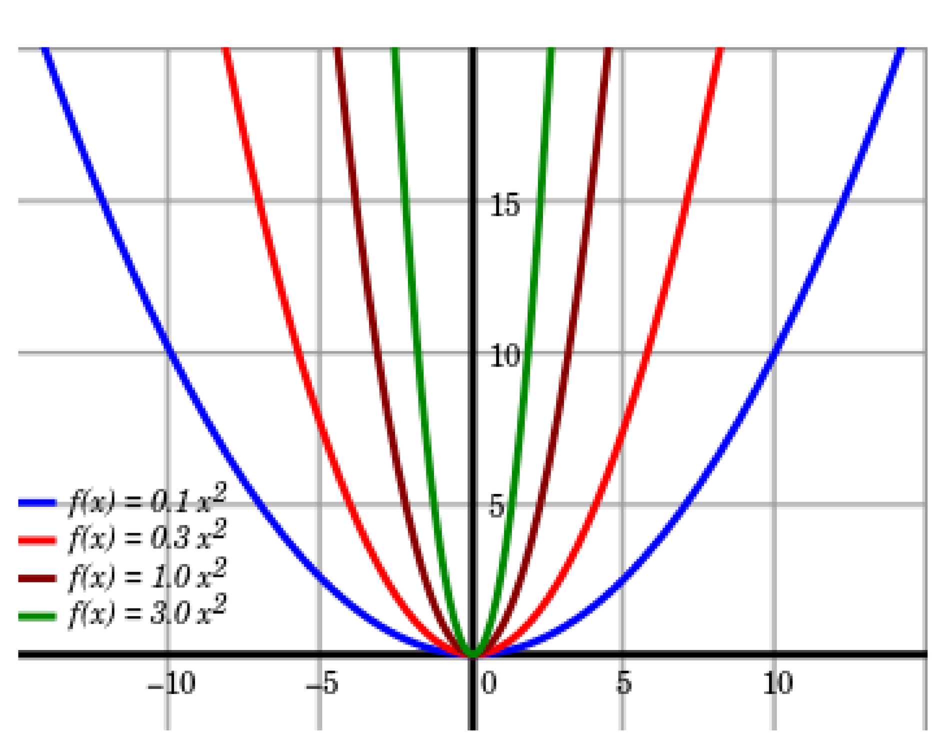

Figure 1: Graph of $f(x) = ax^2|_{\{0.1,0.3,1.0,3.0\}}$

Factorising from JCH

Factorising a quadratic means putting it into two brackets, and is useful if you're trying to draw a graph of a quadratic solve a quadratic equation. Beware a (in ax^2+bx+c form) is not always 1.

In order to factorise a quadratic you should follow steps outlined below:

1. Rearrange the equation into the standard $ax^2 + bx + c$ form.
2. Write down two brackets: $(x \quad)(x \quad)$
3. Find two numbers that multiply to give 'c' and add or subtract to give 'b' (ignoring signs).
4. Put the numbers in brackets and choose their signs.

Factorising- Tasks

1. Factorise $x^2 - x - 12$.

2. Find the two values of k such that $x^2 + kx + 9$ has two equal roots.

3. If $x^2 + (2p - 3q)x + (3p + 2q) = x^2 - 3x + 7$, find p and q

4. IF $ax^2 + bx(x - 4) + c(x - 4) = x^2 + 13x - 20$, find a,b and c

LCH required skills

Long division, definition of root and factor, u shaped or n shaped. Match coefficients even when they are symbols. Recognise a difference of **squares** and **cubes**. A square root has two solutions. Interpret what the value of $b^2 - 4ac$ means for the roots.

Example of Factorisation

Solve $x^2 + 4x - 21 = 0$ by factorising.

$$x^2 + 4x - 21 = (x \qquad)(x \qquad)$$

1 and 21 multiply to give 21 - and add or subtract to give 22 and 20.
3 and 7 multiply to give 21 - and add or subtract to give 10 and 4.

$$x^2 + 4x + 21 = (x + 7)(x - 3)$$

And solving the equation:

$$(x + 7)(x - 3) = 0$$

we get

$$x = -7, \quad x = 3$$

Algebra with roots

Let α and β be the roots of $ax^2 + bx + c$ Let's prove that:

$$\alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Given that α and β are roots, we can write the quadratic as

$$a(x - \alpha)(x - \beta) = ax^2 + bx + c$$

Multiplying out

$$ax^2 - a(\alpha + \beta)x + a\alpha\beta = ax^2 + bx + c$$

and matching coefficients

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

More problems

1. If α and β are roots of
$$x^2 - 9x + 7$$
Find $\frac{1}{\alpha} + \frac{1}{\beta}$ without solving the quadratic!
Find a quadratic whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

2. If $(x + p)^2 - q = x^2 - 4x - 10$ Find p and q

Glossary

verb	noun	meaning
value	plug in	$f(4)$
solve	solution	getting answer
substitute	substitution	$t = x^2$

Table 1: Word Formation

Useful Vocabulary

- (n.) sign $\to$ + or $-$
- (n.) equation $\to$ *something* $= 0$
- (n.) factor $\to$ two multiplied factors give result
- (v.) factorise $\to$ putting into brackets
- (n.) coefficient $\to$ a constant number i.e. a, b, c in a pattern $ax^2 + bx + c$
- (n.) quadratic function $\to$ $f(x) = ax^2 + bx + c$
- (n.) root $\to$ solution of a quadratic or cubic

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Algebra: Cubic Functions

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Objectives

- Understand the concepts of roots and factors.
- Factorising Cubics.
- Proving some useful identities.
- Use the $-b$ formula yet again!

Review

Form of a Cubic Function

- $f(x) = ax^3 + bx^2 + cx + d$ is form of a cubic.
- $f(x) = a(x - x_1)(x - x_2)(x - x_3)$ is called the **factored form**, where x_1, x_2 and x_3 are the roots of the cubic function. These are the values of x where we cut the x axis.
- We can also express as $f(x) = a(x - x_1)(x^2 + rx + s)$ This is frequently examined.

What cubics look like

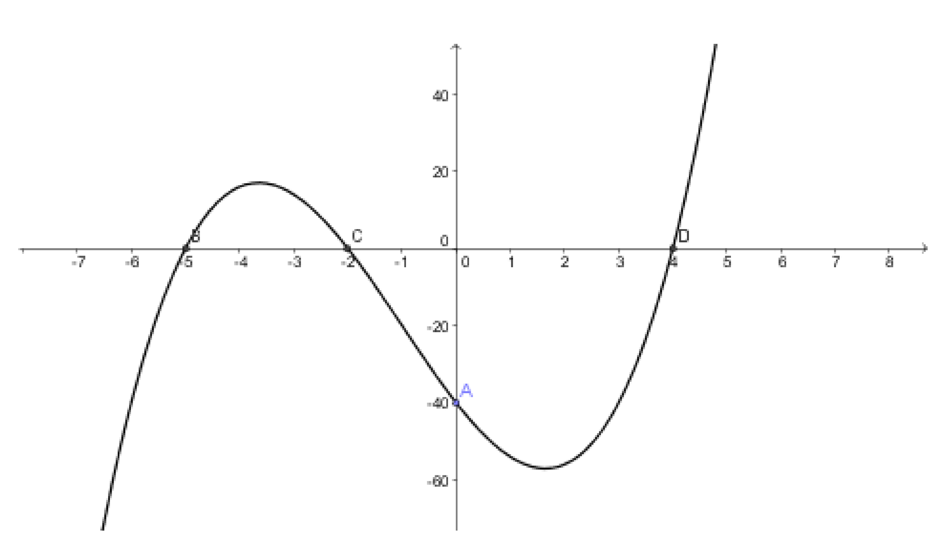

Figure 1: Graph of a Cubic with a positive

Factorising a Cubic

Factorising a cubic means putting it into three brackets, and is useful if you're trying to draw a graph . Beware a (in $ax^3 + bx^2 + cx + d$ form) is not always 1.

In order to factorise a cubic you should follow steps outlined below:

1. For LCH, one root will **always** be an **integer**
2. Write down the first factor $x - x_1$
3. Perform long division to get a quadratic
4. Use $-b$ formula to get two remaining roots

LCH Problems Paper 1

1. **2017** Factorise $2x^3 + 5x^2 - 4x - 3$ given that $x = -3$ is a root.

2. **2015** Solve $x^3 - 3x^2 - 9x + 11 = 0$

3.**2014** A function $f(x)$ satisfies $f(-3) = 0$, $f(-1) = 0$ and $f(2) = 0$. It cuts the y axis at $(0, -6)$. Verify $f(x)$ can be written as $x^3 + 2x^2 - 5x - 6$

LCH required skills

Long division, definition of root and factor, what shape. Match coefficients even when they are symbols. Recognise a difference of **squares** and **cubes**. A square root has two solutions. Interpret what the value of $b^2 - 4ac$ means for the roots of the quadratic remainder after long division

LCH Paper 1 2018 question 2

$f(x) = x^3 - 17x + 80x - 64$. Show that $f(1) = 0$ and find **another** root of $f(x)$.

Solution $x = 1$ is a root, so $(x - 1)$ is a factor. Why? Divide $x - 1$ into $f(x)$

$$\begin{array}{r} x^2 - 16x + 64 \\ x-1 \overline{) x^3 - 17x^2 + 80x - 64} \\ -x^3 \quad + x^2 \qquad\qquad \\ \hline -16x^2 + 80x \qquad \\ 16x^2 - 16x \qquad \\ \hline 64x - 64 \\ -64x + 64 \\ \hline 0 \end{array}$$

LCH 2018 continued

We now have a quadratic remainder. By inspection, we can see that it has two equal roots $x_2 = 8$ and x_3=8. The factorisation will be

$$f(x) = (x - 1)(x - 8)(x - 8)$$

We can use the $-b$ formula but also can you show that $b^2 - 4ac = 0$? This is the condition for equal roots!

Calculus Problems

1. Find the critical points of

$$x^3 + 2x^2 - 5x - 6$$

Determine which critical point is a maximum, minimum and point of inflection.

2. If $(x+p)^2 - q = x^2 - 4x - 10$. Find p and q. Hence determine the minimum point of the function. Why is a minimum and not a maximum? What domain is the function increasing and decreasing?

Glossary

verb	noun	meaning
value	plug in	$f(4)$
solve	solution	getting answer
substitute	substitution	$t = x^2$

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- (n.) cubic $\to$ $f(x) = ax^3 + bx^2 + cx + d$
- (n.) root $\to$ solution of a quadratic or cubic

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Algebra: Tricks for LCH

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Objectives

- Develop several algebra tricks to make exam easier
- Identify method required in context of question
- See that the square of any **real** complicated algebraic expression is always positive
- Get better with cubics

Review

There are only 5-6 algebra question types that basically repeat themselves.

Simultaneous Equations

Algebraic Inequalities

Algebraic Fractions

Quadratics

Cubics

We have learned to tackle these questions but sometimes, success in solution depends on a few subtle little algebra tools. ** denote tough problems or concepts.

Difference of Two Squares

Factorising, for LCH can mean simplifying. Best with examples. Maybe the most challenging? All **Paper 1**

Factorize $x^2 - 16$ **Solution** $(x-4)(x+4)$

Factorize $3x^2 - 75$ **Solution** $3(x^2-25) = 3(x-5)(x+5)$

Factorize $x - \frac{1}{x}$ **Solution** $(\sqrt{x} - \frac{1}{\sqrt{x}})(\sqrt{x} + \frac{1}{\sqrt{x}})$

Simplify $\frac{x-9}{\sqrt{x}+3}$ **Solution** $\frac{(\sqrt{x}+3)(\sqrt{x}-3)}{\sqrt{x}+3} = \sqrt{x} - 3$

LCH required skills

Factors, multiples. Difference of squares:$x - y = (\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})$! Difference of cubes. Conditions on roots of quadratic to be real distinct, real equal or complex numbers. Finding common denominators for symbols rather than numbers. Binomial Theorem**

Tricks for Algebaic Fractions

Most important observation from class is

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}$$

2007** Show, if $x + a \geq 0$,

$$x + \frac{9}{x+a} \geq 6 - a$$

We can multiply across by $x+a$. Why? You should end up trying to prove that

$$x^2 + 2ax - 6x + a^2 - 6a + 9 \geq 0$$

This can be rearranged to try and prove

$$x^2 + 2(a-3)x + (a-3)^2 \geq 0$$

but this is

$$(x + (a-3))^2 \geq 0$$

which is always true. Ugh!!!

Difference of Two Cubes

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

$$x^3 + y^3 = (x+y)(x^2-xy+y^2)$$

Problem If $\alpha + \beta = 4$ and $\alpha\beta = -5$ find value of $\alpha^3 + \beta^3$?

Solution $(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2) = 4(\alpha^2 + \beta^2 + 5)$

But $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 16 + 10 = 26$

So solution is 124!

Tricks for Algebraic Inequalities

Observe that $(a-b)^2$ is positive for all $a, b \epsilon R$ If we can get an algebraic expression in this form, we can prove general inequalities.

2005 show that for all a and b , reals

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

solution Square both sides and simplify to get

$$a^2 + 2ab + b^2 \leq 2(a^2+b^2)$$

REARRANGE

$$a^2 - 2ab + b^2 \geq 0$$

and we are done. Why?

**Binomial Theorem

We can multiply out $(x+y)^3$ to get

$$x^3 + 3x^2y + 3xy^2 + y^3$$

But what about $(x+y)^8$? There is a lovely result called the binomial theorem that gives this to you. It uses same mathematics from probability called combinations.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}x^2y^{n-1} + \binom{n}{n}y^n$$

The general term is

$$\binom{n}{r}x^{n-r}y^r$$

where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

2018 Paper 2

$$\frac{(n+3)!(n+2)!}{(n+1)!(n+1)!} = ax^3 + bx^2 + cx + d$$

Find a,b,c and d.

2005 The first three terms in the binomial expansion of $(1+kx)^n$ are $1 - 21x + 189x^2$. Find the value of k and n.

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Exponentials and Logarithms

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Objectives

- Underpinning of all of multiplication and division
- Understand that log is the inverse of exponential
- Solve logarithm algebra questions
- Learn tricks vital to whole exam

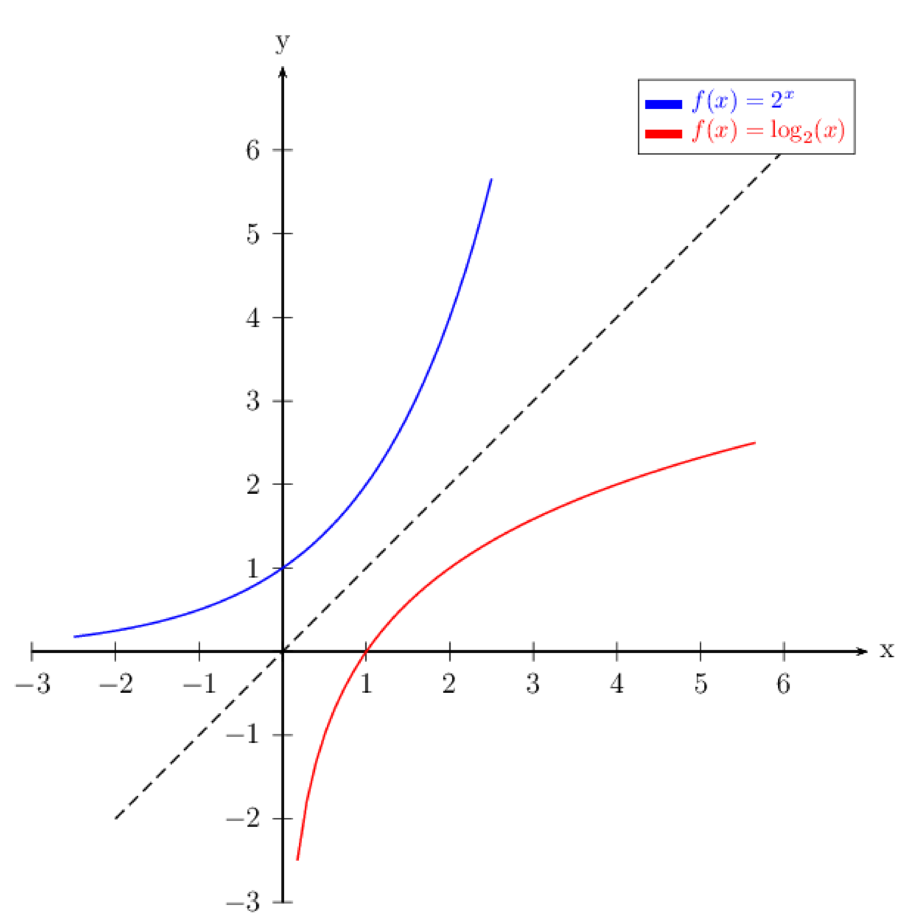
Review

Exponentials

- Expression of the form a^p ,a is the base
- We can multiply $a^p a^q = a^{p+q}$
- Take the power $(a^p)^q = a^{pq}$
- Two bases, $(ab)^p = a^p b^q$
- $\sqrt{6} = 6^{\frac{1}{2}} = 2^{\frac{1}{2}}3^{\frac{1}{2}} = \sqrt{2}\sqrt{3}$
- We use these definitions to define fractions $a^{-1} = \frac{1}{a^1} = \frac{1}{a}$.

Logarithms

- Inverse of exponentials
- $log_a(xy) = log_a x + log_a y$
- $log_a(\frac{x}{y}) = log_a x - log_a y$
- $log_a x^n = n log_a x$
- $log_a a^n = n$
- $log_3 729 = 6$



Exponentials and Algebra, again

All **Paper 1**

2008 Solve $2^{x^2} = 8^{2x+9}$ **Solution** $2^{x^2} = 2^{6x+27}$. So $x^2 = 6x + 27$ giving as factorization

$$(x-9)(x+3)$$

2011 Solve $3^{2x+1} - 17(3^x) - 6 = 0$ **Solution** Let $y = 3^x$. Can you show above equation reduces to

$$3y^2 - 17y - 6 = 0$$

factorization of which is $(3y+1)(y-6)$ so $3^x = -3^{-1}$ or $3^x = 6$. First root does not exist as an exponential can never be negative. So $x = \frac{ln6}{ln3}$

Logarithms and Algebra

2016 Given $log_a 2 = p$ and $log_a 3 = q$ express in terms of p and q

$$log_a \frac{8}{3}$$

and

$$log_a \frac{9a^2}{16}$$

Solution

$$log_a \frac{8}{3} = log_a 8 - log_a 3 = 3p - q$$

$$log_a \frac{9a^2}{16} = log_a 9a^2 - log_a 16$$

which reduces to

$$2q + 2 - 4p$$

Logarithm inverse of Exponential

If $a^x = y$ then $x = log_a y$

Further Indices Problems

1995
Solve $2^x + 2^{1-x} - 3 = 0$
2003
Solve $2^{2y+1} - 5(2^y) + 2 = 0$
2002
Solve $\frac{8}{2^x} = 32$
2000
Solve $3e^x - 7 + 2e^{-x} = 0$
Important observation: Comes up in calculus questions often:
Solve for x

$$(x^2-1)e^{-x^2} = 0$$

NOTE that exponential functions never return zero or a negative number. So only solutions are when $(x^2-1) = 0$ yielding either $x = 1$ or $x = -1$

Further Log Problems

1995
Solve $log_2(x+2) + log_2(x-2) = 5$
2000
Solve $2log_9 x = \frac{1}{2} + log_9(5x+18)$
2004
Solve $log_4 x - log_4(x-2) = \frac{1}{2}$
1996**
Solve the simultaneous equations

$$log(x+y) = 2log(x)$$
$$log(y) = log(2) + log(x-1)$$

This looks difficult but reduces to

$$x + y = x^2$$

and

$$y = 2(x-1)$$

for which we solve the quadratic $x^2 - 3x + 2$

Important Result for Logs

2005**

$$log_a b = \frac{1}{log_b a}$$

Need to use inverse. Let $log_a b = x$. Then $b = a^x$ by definition. So,

$$a = b^{\frac{1}{x}}$$

and

$$\frac{1}{x} = log_b a$$

Voila!
1999**
Show that

$$\frac{1}{log_2 x} + \frac{1}{log_3 x} + \frac{1}{log_5 x} = \frac{1}{log_{30} x}$$

Using above general result, this reduces to

$$log_x 2 + log_x 3 + log_x 5$$

which equals $log_x 30$ and

$$log_x 30 = \frac{1}{log_{30} x}!$$

Easy to Forget!

- a^x positive for all $x \epsilon R$
- $log_a x$ only defined for $x > 0$
- $log_a a = 1$
- $log_a \sqrt{a} = \frac{1}{2}$
- $2log_a(x) = log_a(x^2)$

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Leaving Certificate

Project Mathematics Higher

Module One: Algebra

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Introduction to Algebra

This document is the first module in a sequence of modules which will cover all sections of the LC Project Mathematics syllabus. We will start with algebra which is the backbone of the course. We will use algebra style questions from Projects Mathematics papers since the new syllabus was introduced. The issue confronting students is there is not enough material reflecting the range of possible questions from the syllabus. Theory will be interspersed with the exam questions as we progress, so we will learn the theory from doing questions rather than the other way round! Another issue tutoring project maths is that any one question can combine techniques from several parts of the course. Some of the examples that follow may only be partial questions as the subsequent part may involve calculus. But mastering algebra will make your calculus really excel.

The structure of this module and the subsequent modules will be Section one, Project Mathematics questions. Section two, pre Project Mathematics questions, all interspersed with theory. There may be appendices pointing out more esoteric ideas. Any theory that is non-examinable will be highlighted by an asterisk.

Project Mathematics

Question 1

(a) Solve the simultaneous equations.

$$\begin{aligned} 2x + 3y - z &= -4 \\ 3x + 2y + 2z &= 14 \\ x \qquad - 3z &= -13 \end{aligned}$$

(b) Solve the inequality $\frac{2x-3}{x+2} \geq 3$, where $x \in \mathbb{R}$ and $x \neq -2$.

Question 2 **(25 marks)**

(a) The amount of a substance remaining in a solution reduces exponentially over time. An experiment measures the percentage of the substance remaining in the solution. The percentage is measured at the same time each day. The data collected over the first 4 days are given in the table below. Based on the data in the table, estimate which is the first day on which the percentage of the substance in the solution will be less than 0·01%.

Day	**1**	**2**	**3**	**4**
Percentage of substance (%)	95	42·75	19·2375	8·6569

Quadratics

There are three main polynomials on the course, the line (the exam likes to use l and k to name linear functions):

$$l(x) = mx + c$$

where m is the slope and c is where we have the line crossing the y axis.
We then have the quadratic:

$$f(x) = ax^2 + bx + c$$

and the cubic:

$$g(x) = ax^3 + bx^2 + cx + d$$

For quadratics we have two roots which I prefer to call α and β as we discussed in class:

$$\alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

The expression inside the square root determines the nature of the roots. For two distinct roots which are real numbers:

$$b^2 - 4ac > 0$$

For equal, but real roots:

$$b^2 - 4ac = 0$$

For complex number roots, which we will cover in a later module:

$$b^2 - 4ac < 0$$

We do not need to know the values of the roots but we can show that

$$\alpha + \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} + (\frac{-b + \sqrt{b^2 - 4ac}}{2a})$$

$$= \frac{-b}{2a} + (\frac{-b}{2a})$$

$$= \frac{-b}{a}$$

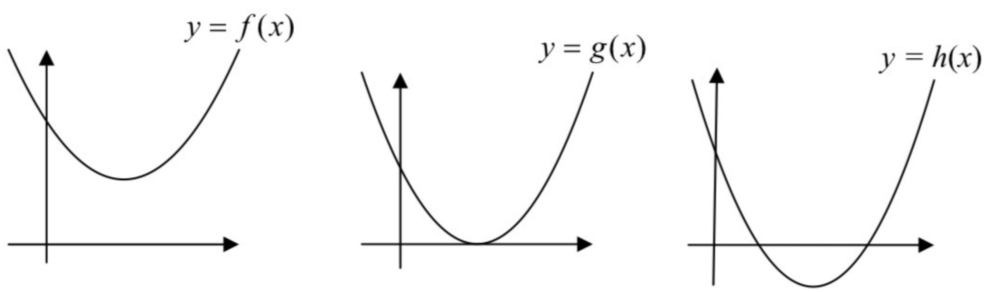
Using $x^2 - y^2 = (x - y)(x + y)$, we can show (try yourselves), that

$$\alpha\beta = (\frac{-b - \sqrt{b^2 - 4ac}}{2a})(\frac{-b + \sqrt{b^2 - 4ac}}{2a})$$

$$= \frac{c}{a}$$

These identities are useful to solve pre Project Mathematics questions but have not being taken off the syllabus. Also they are useful for improving algebra skills.

The graphs of three quadratic functions, f, g and h, are shown.



In each case, state the nature of the roots of the function.

Question 3

The function f is such that $f(x) = 2x^3 + 5x^2 - 4x - 3$, where $x \in \mathbb{R}$.

(a) Show that $x = -3$ is a root of $f(x)$ **and** find the other two roots.

If you're given a cubic, always assume one root is an integer

$x = -3$ is a root, so $(x+3)$ is a factor

$$
\begin{array}{r|l}
 & 2x^2 - x - 1 \\
x+3 & 2x^3 + 5x^2 - 4x - 3 \\
 & 2x^3 + 6x^2 \\
 & \quad -x^2 - 4x \\
 & \quad -x^2 - 3x \\
 & \qquad -x - 3 \\
 & \qquad -x - 3 \\
 & \qquad 0x + 0
\end{array}
$$

note if divides in evenly must have zero remainder

So $2x^2 - x - 1$

$= (2x+1)(x-1)$

$x = -3$
$x = 1$
$x = -\frac{1}{2}$

Exponential Functions

Funtions which grow very quickly are called exponentials

$$f(x) = 2^x$$

as the classic example. We can generalise this to

$$g(x) = a^x$$

where a can be any number. We recall $a = 10$ from Junior Certificate. These exponentials lead to logarithms which are just inverses of these. These will be discussed in a later module.

By far the most important exponential is e^x where the number e is a bit like π. These numbers are like $\sqrt{2}$ in that they are irrational and cannot be expressed as a fraction, or as a rational number.

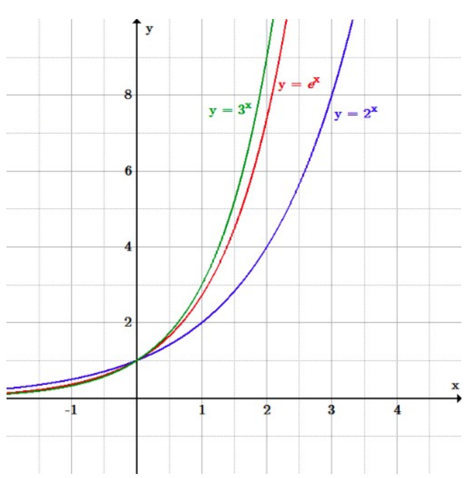
e is approximately 2.7182... but never repeats unlike $\frac{1}{3}$.

(Non-examinable

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}$$

)

So, if you graph 2^x and 3^x, e^x will lie between them. e^x is a function on your calculator. This function appears everywhere on the exam.



Question 4

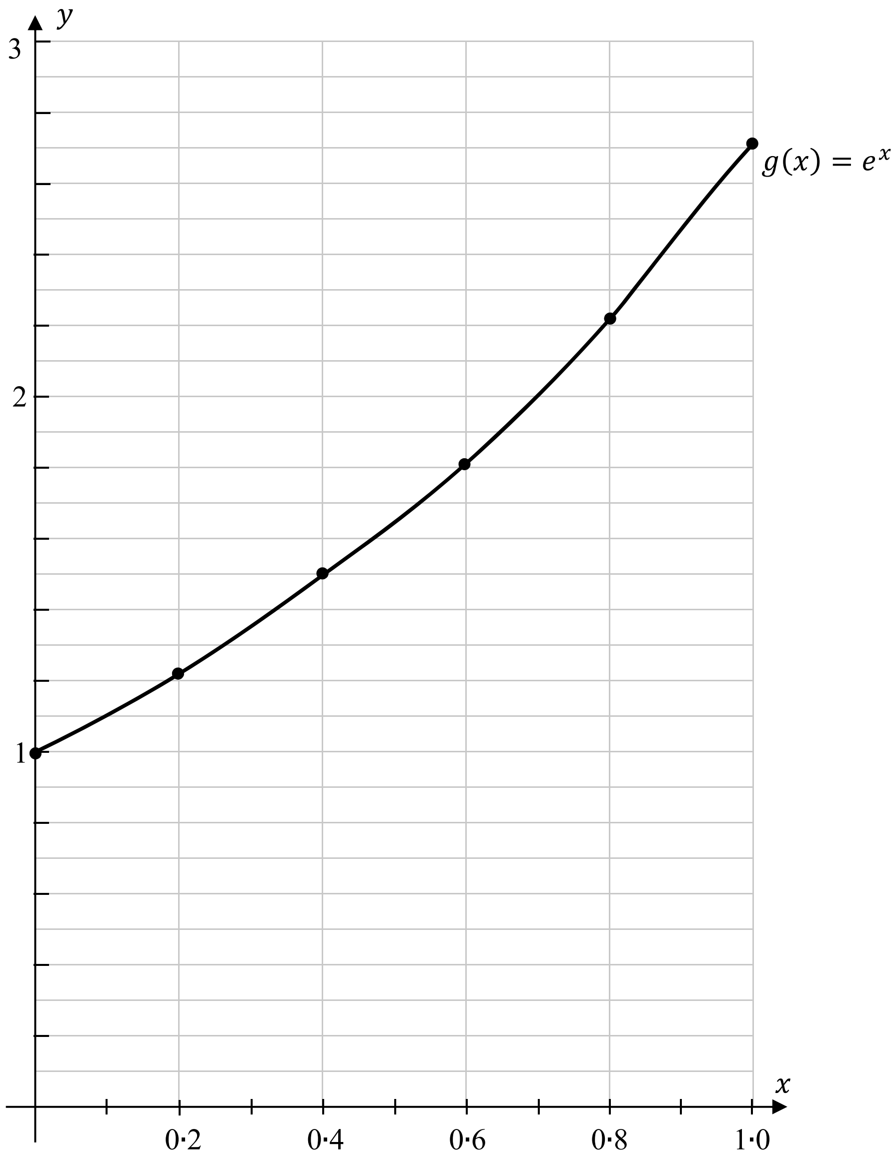
The graph of the function $g(x) = e^x$, $x \in \mathbb{R}$, $0 \leq x \leq 1$, is shown on the diagram below.

(a) On the same diagram, draw the graph of $h(x) = e^{-x}$, $x \in \mathbb{R}$, in the domain $0 \leq x \leq 1$.

Recall

$a^x = \frac{1}{a^{-x}}$.

so $e^{-x} = \frac{1}{e^x}$



Question 5

Sometimes it is possible to predict the future population in a city using a function.
The population in Sapphire City, over time, can be predicted using the following function:

$$p(t) = Se^{0\cdot 1t} \times 10^6.$$

The population in Avalon, over time, can be predicted using the following function:

$$q(t) = 3\cdot 9e^{kt} \times 10^6.$$

In the functions above, t is time, in years; $t = 0$ is the beginning of 2010; and both S and k are constants.

(a) The population in Sapphire City at the beginning of 2010 is 1 100 000 people.
Find the value of S.

(b) Find the predicted population in Sapphire City at the beginning of 2015.

(c) Find the predicted change in the population in Sapphire City during 2015.

(d) The predicted population in Avalon at the beginning of 2011 is 3 709 795 people.
Write down and solve an equation in k to show that $k = -0{\cdot}05$, correct to 2 decimal places.

(e) Find the year during which the populations in both cities will be equal.

(f) Find the predicted average population in Avalon from the beginning of 2010 to the beginning of 2025.

Question 6

Solve the equation $x^3 - 3x^2 - 9x + 11 = 0$.
Write any irrational solution in the form $a + b\sqrt{c}$, where $a, b, c \in \mathbb{Z}$.

If not told, always one root of a cubic will be an integer.
Here obviously $x = 1$ works. But you may have to try $-1, -2, -3$ or $1, 2, 3$

So $(x-1)$ is a factor:

$$\begin{array}{r|l} & x^2 - 2x - 11 \\ x-1 & x^3 - 3x^2 - 9x + 11 \\ & x^3 - x^2 \\ & -2x^2 - 9x \\ & -2x^2 + 2x \\ & -11x + 11 \\ & -11x + 11 \\ & 0x + 0 \end{array}$$

You "$-b$" formula to find roots of $x^2 - 2x - 11$

$a = 1$
$b = -2$
$c = -11$

$$\frac{2 \pm \sqrt{4 + 44}}{2} = 1 \pm \frac{\sqrt{48}}{2} = 1 \pm 2\sqrt{3}$$

Question 7

Let $f(x) = -x^2 + 12x - 27,\ x \in \mathbb{R}$.

(a) **(i)** Complete Table 1 below.

Table 1							
x	3	4	5	6	7	8	9
$f(x)$	0	5			8		

(ii) Use Table 1 and the trapezoidal rule to find the approximate area of the region bounded by the graph of f and the x-axis.

9

3

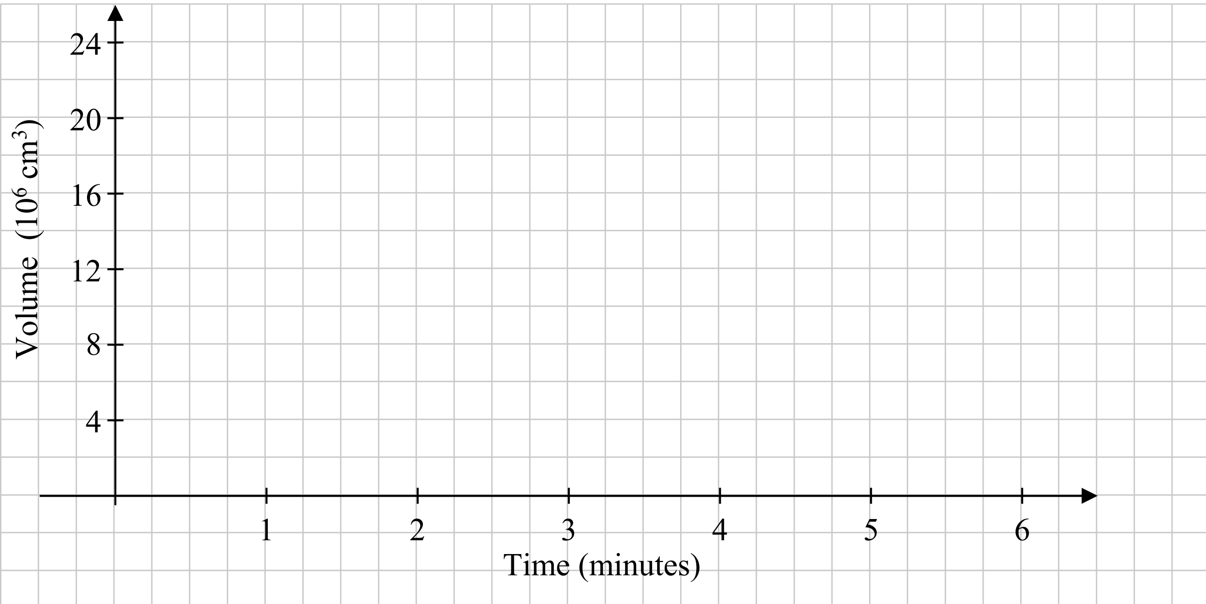
Question *8*

An oil-spill occurs off-shore in an area of calm water with no currents. The oil is spilling at a rate of 4×10^6 cm^3 per minute. The oil floats on top of the water.

(a) **(i)** Complete the table below to show the total volume of oil on the water after each of the first 6 minutes of the oil-spill.

Time (minutes)	1	2	3	4	5	6
Volume (10^6 cm^3)		8				

(ii) Draw a graph to show the total volume of oil on the water over the first 6 minutes.



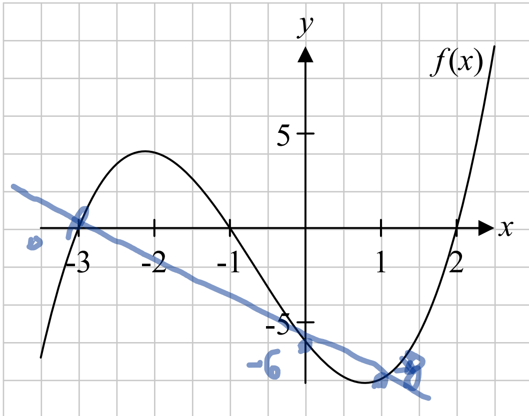
(iii) Write an equation for $V(t)$, the volume of oil on the water, in cm^3, after t minutes.

(b) The spilled oil forms a circular oil slick **1 millimetre** thick.

(i) Write an equation for the volume of oil in the slick, in cm^3, when the radius is r cm.

Question 9

(a) The graph of a cubic function $f(x)$ cuts the x-axis at $x = -3$, $x = -1$ and $x = 2$, and the y-axis at $(0, -6)$, as shown.



Verify that $f(x)$ can be written as

$$f(x) = x^3 + 2x^2 - 5x - 6.$$

So $(x+3)$, $(x+1)$ and $(x-2)$ are factors.

$f(x) = (x+3)(x+1)(x-2)$

$= (x+3)(x^2 - x - 2)$

$= (x^3 - x^2 - 2x + 3x^2 - 3x - 6) = x^3 + 2x^2 - 5x - 6$

(b) **(i)** The graph of the function $g(x) = -2x - 6$ intersects the graph of the function $f(x)$ above. Let $f(x) = g(x)$ and solve the resulting equation to find the co-ordinates of the points where the graphs of $f(x)$ and $g(x)$ intersect.

$g(x)$ is a line : why? $m = -2$. $c = -6$.

So : $g(x) = f(x)$ should be points of intersection of the line with the cubic :

$-2x - 6 = x^3 + 2x^2 - 5x - 6$

$\Rightarrow x^3 + 2x^2 - 3x + 0 = 0$.

$= x(x^2 + 2x - 3) = 0$

So $x = 0$, $x = -3$, $x = +1$

(ii) Draw the graph of the function $g(x) = -2x - 6$ on the diagram above.

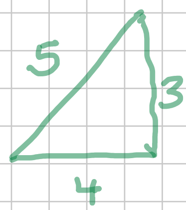
Question 10) (a)

Three natural numbers a, b and c, such that $a^2 + b^2 = c^2$, are called a Pythagorean triple.

(i) Let $a = 2n+1$, $b = 2n^2 + 2n$ and $c = 2n^2 + 2n + 1$.
Pick one natural number n and verify that the corresponding values of a, b and c form a Pythagorean triple.

Try $n = 1$

$a = 3$, $b = 4$, $c = 5$



(ii) Prove that $a = 2n+1$, $b = 2n^2 + 2n$ and $c = 2n^2 + 2n + 1$, where $n \in \mathbb{N}$, will always form a Pythagorean triple.

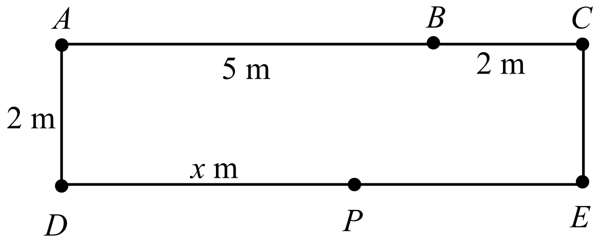
$a^2 = 4n^2 + 4n + 1$ $\quad b^2 = 4n^4 + 8n^3 + 4n^2$

$$c^2 = (2n^2+2n+1)(2n^2+2n+1)$$
$$= 4n^4 + 4n^3 + 2n^2 + 4n^3 + 4n^2 + 2n + 2n^2 + 2n + 1$$
$$= 4n^4 + 8n^3 + 8n^2 + 4n + 1$$

But $a^2 + b^2 = 4n^4 + 8n^3 + 8n^2 + 4n + 1$

So indeed the case for all $n \in \mathbb{N}$

(b) *ADEC* is a rectangle with $|AC| = 7$ m and $|AD| = 2$ m, as shown.
B is a point on $[AC]$ such that $|AB| = 5$ m.
P is a point on $[DE]$ such that $|DP| = x$ m.



(i) Let $f(x) = |PA|^2 + |PB|^2 + |PC|^2$.

Show that $f(x) = 3x^2 - 24x + 86$, for $0 \le x \le 7$, $x \in \mathbb{R}$.

Question 11

Ciarán is preparing food for his baby and must use cooled boiled water. The equation $y = Ae^{kt}$ describes how the boiled water cools. In this equation:

- t is the time, in minutes, from when the water boiled,
- y is the *difference* between the water temperature and room temperature at time t, measured in degrees Celsius,
- A and k are constants.

The temperature of the water when it boils is $100°C$ and the room temperature is a constant $23°C$.

(a) Write down the value of the temperature difference, y, when the water boils, and find the value of A.

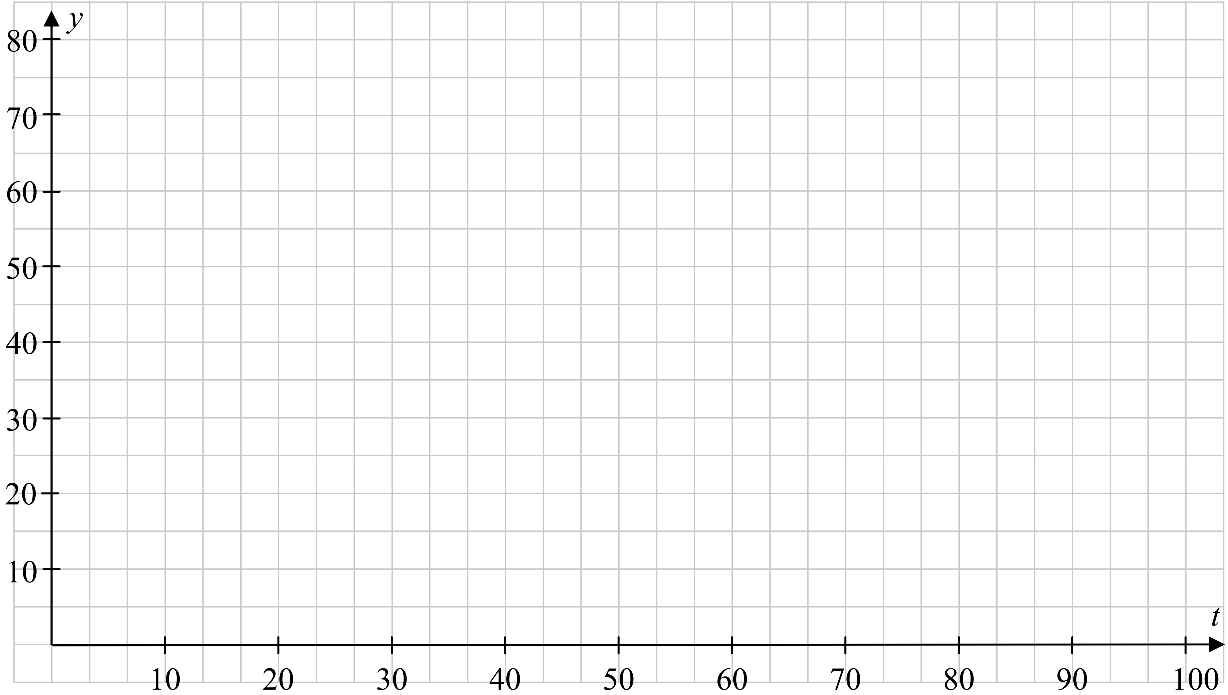
$y =$

$A =$

(b) After five minutes, the temperature of the water is $88°C$.
Find the value of k, correct to three significant figures.

(c) Ciarán prepares the food for his baby when the water has cooled to $50°C$. How long does it take, correct to the nearest minute, for the water to cool to this temperature?

(d) Using your values for A and k, sketch the curve $f(t) = Ae^{kt}$ for $0 \le t \le 100$, $t \in \mathbb{R}$.



(e) **(i)** On the same diagram, sketch a curve $g(t) = Ae^{mt}$, showing the water cooling at a *faster* rate, where A is the value from part **(a)**, and m is a constant. Label each graph clearly.

(ii) Suggest one possible value for m for the sketch you have drawn and give a reason for your choice.

Question 12

(a) Find the set of all real values of x for which $2x^2 + x - 15 \geq 0$.

(b) Solve the simultaneous equations;

$$x + y + z = 16$$

$$\frac{5}{2}x + y + 10z = 40$$

$$2x + \frac{1}{2}y + 4z = 21.$$

Question 13

Scientists can estimate the age of certain ancient items by measuring the proportion of carbon−14, relative to the total carbon content in the item. The formula used is $Q = e^{-\frac{0\cdot693t}{5730}}$, where Q is the proportion of carbon−14 remaining and t is the age, in years, of the item.

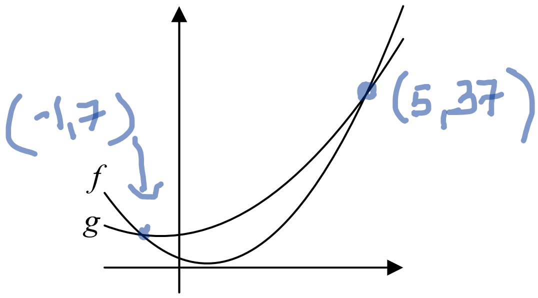
(a) An item is 2000 years old. Use the formula to find the proportion of carbon−14 in the item.

(b) The proportion of carbon−14 in an item found at Lough Boora, County Offaly, was 0·3402. Estimate, correct to two significant figures, the age of the item.

Question *14* **(25 marks)**

The functions *f* and *g* are defined for $x \in \mathbb{R}$ as

$$f : x \mapsto 2x^2 - 3x + 2 \quad \text{and}$$
$$g : x \mapsto x^2 + x + 7 .$$



(a) Find the co-ordinates of the two points where the curves $y = f(x)$ and $y = g(x)$ intersect.

This occurs when $f(x) = g(x)$

$$2x^2 - 3x + 2 = x^2 + x + 7 .$$
$$x^2 - 4x - 5 = 0$$
$$(x-5)(x+1) = 0$$

So $x = 5$ and $x = -1$.

$y = 25 + 5 + 7 = 37$ at $x = 5$.

and $y = 1 - 1 + 7 = 7$ at $x = -1$

Question *15* **(25 marks)**

A cubic function f is defined for $x \in \mathbb{R}$ as

$$f : x \mapsto x^3 + (1-k^2)x + k, \quad \text{where } k \text{ is a constant.}$$

(a) Show that $-k$ is a root of f.

let $x = -k$, then $f(-k)$ will be
$-k^3 + (1-k^2)(-k) + k$.
$= -k^3 - k + k^2 + k = 0$

(b) Find, in terms of k, the other two roots of f.

$(x+k)$ is a factor, so realising there is no x^2 term.

$$
\begin{array}{r|l}
 & x^2 - kx + 1 \\
x+k & x^3 + 0x^2 + (1-k^2)x + k \\
 & x^3 + kx^2 \\
 & -kx^2 + (1-k^2)x \\
 & -kx^2 - k^2x. \\
 & x + k. \\
 & x + k. \\
 & 0x + 0.
\end{array}
$$

So other 2 roots are

$$\frac{k \pm \sqrt{k^2-4}}{2}$$

(c) Find the set of values of k for which f has exactly one real root.

This will be when $k^2 - 4 = 0$
$(b^2 - 4ac = 0)$. So $k^2 = 4$
when $k = +2$
or $k = -2$.

Question *16*

(a) Solve the simultaneous equations,

$$2x + 8y - 3z = -1$$
$$2x - 3y + 2z = 2$$
$$2x + y + z = 5.$$

$2x + 8y - 3z = -1$
$2x - 3y + 2z = 2$
$11y - 5z = -3$

$2x - 3y + 2z = 2$
$2x + y + z = 5$
$-4y + z = -3.$

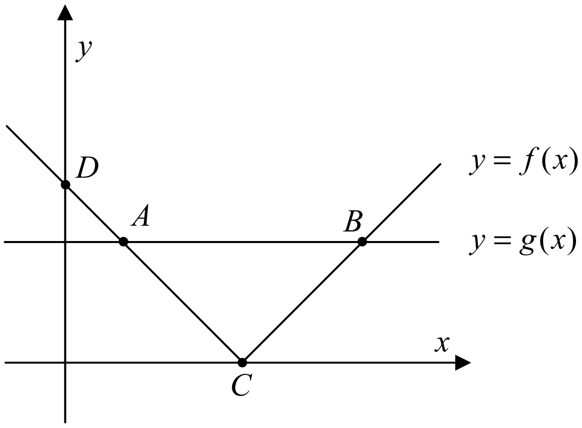
$11y - 5z = -3$
$-4y + z = -3. \quad (\times 5)$

$11y - 5z = -3$
$-20y + 5z = -15.$
$-9y = -18, \quad y = 2.$
$z = 5.$
$x = -1$

(b) The graphs of the functions $f : x \mapsto |x - 3|$ and $g : x \mapsto 2$ are shown in the diagram.

(i) Find the co-ordinates of the points *A*, *B*, *C* and *D*.

$|x - 3| = 2$ when $x = 5$ and $x = 1$



$A = (1, 2)$ $\quad B = (5, 2)$

$C = (3, 0)$ $\quad D = (0, 3)$

(ii) Hence, or otherwise, solve the inequality $|x - 3| < 2$.

Well, can read from the graph?
$1 < x < 5.$